Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series B 97 (2007) 1074–1076
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Note

Perfect matchings extend to Hamilton cycles in hypercubes

Jiří Fink

Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague 1, Czech Republic

Received 13 December 2006

Available online 12 March 2007

Abstract

Kreweras' conjecture [G. Kreweras, Matchings and Hamiltonian cycles on hypercubes, Bull. Inst. Combin. Appl. 16 (1996) 87–91] asserts that any perfect matching of the hypercube Q_d, $d \geqslant 2$, can be extended to a Hamilton cycle. We prove this conjecture.

Keywords: Hypercube; Hamiltonian cycle; Perfect matching

1. Introduction

A set of edges $P \subset E$ of a graph $G = (V, E)$ is a *matching* if every vertex of G is incident with at most one edge of P. If a vertex v of G is incident with an edge of P, we say that v is *covered* by P. A matching P is *perfect* if every vertex of G is covered by P.

The d-dimensional hypercube Q_d is a graph whose vertex set consists of all binary vectors of length d, with two vertices being adjacent whenever the corresponding vectors differ at exactly one coordinate.

It is well known that Q_d is Hamiltonian for every $d \geqslant 2$. This statement can be traced back to 1872 [3]. Since then the research on Hamilton cycles in hypercubes satisfying certain additional properties has received considerable attention. An interested reader can find more details about this topic in the survey of Savage [2], e.g. Dvořák [4] showed that any set of at most $2d - 3$ edges of Q_d ($d \geqslant 2$) that induces vertex-disjoint paths is contained in a Hamilton cycle. Dim-

E-mail address: fink@atrey.karlin.mff.cuni.cz.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2007.02.007

itrov et al. [5] proved that for every perfect matching P of Q_d ($d \geqslant 3$) there exists a Hamilton cycle that faults P if and only if P is not a layer of Q_d.

Kreweras [1] conjectured the following:

Conjecture. *Every perfect matching in the d-dimensional hypercube with $d \geqslant 2$ extends to a Hamilton cycle.*

We prove this conjecture.

Since the 2-dimensional hypercube has only one perfect matching and the 3-dimensional hypercube has two distinct perfect matchings up to isomorphism, the conjecture is easy to check for $d = 2$ and $d = 3$. Kreweras [1] proved the conjecture for $d = 4$.

2. Proof of the conjecture

Let us consider a perfect matching P of the hypercube Q_d which is contained in a Hamilton cycle C of Q_d. Let R denote the set of edges contained in the cycle C but not in P. Obviously, R is also a perfect matching of Q_d. Hence, Kreweras' conjecture can be restated in the following way:

For every perfect matching P of the hypercube Q_d, $d \geqslant 2$, there exists a perfect matching R such that $P \cup R$ is a Hamilton cycle of Q_d.

In fact, we prove the following theorem which is clearly stronger than the above and so implies Kreweras' conjecture. Let K_{Q_d} be the complete graph on the vertices of the hypercube Q_d.

Theorem. *For every perfect matching P of K_{Q_d} there exists a perfect matching R of Q_d, $d \geqslant 2$, such that $P \cup R$ is a Hamilton cycle of K_{Q_d}.*

The crucial step of our proof lies in the following lemma. A forest is *linear*, if each component of it is a path.

Lemma. *Let P be a matching of K_{Q_d} that is not perfect. Then, there exists a perfect matching R of Q_d, $d \geqslant 2$, such that $P \cap R = \emptyset$ and $P \cup R$ is a linear forest.*

Before proving the lemma, let us first prove the theorem.

Proof of Theorem. Let $e = xy$ be an arbitrary edge of P. For the matching $P' = P \setminus \{e\}$, by Lemma, there exists a perfect matching R of Q_d such that $P' \cap R = \emptyset$ and $P' \cup R$ is a linear forest. If $e \in R$ then every vertex of the graph with edge set $(P' \cup R) \setminus \{e\}$ has even degree, but $P' \cup R$ is a forest. Hence, $e \notin R$. Now, it easily follows that $P' \cup R$ is a Hamilton path of K_{Q_d} from x to y. Hence, $P \cup R$ is a Hamilton cycle of K_{Q_d}. $\square$

Now, we prove the lemma:

Proof of Lemma. The proof proceeds by induction on d. The statement holds for $d = 2$. Let us suppose that the statement is true for every hypercube Q_k with $2 \leqslant k \leqslant d - 1$ and let us prove it for d.

We know that P is a matching which is not perfect. Hence, there must exist at least two vertices $u_1, u_2 \in V(Q_d)$ uncovered by P. We can divide the d-dimensional hypercube Q_d into two $(d-1)$-dimensional sub-hypercubes Q^1 and Q^2 such that $u_i \in V(Q^i)$ for $i \in \{1,2\}$. Let $K^i = (V(Q^i), \binom{V(Q^i)}{2})$ and $P^i = P \cap E(K^i)$ for $i \in \{1,2\}$.

The set of edges P^1 is a matching of K^1 which is not perfect since u_1 is not covered. Hence, there exists a perfect matching R^1 of Q^1 such that $R^1 \cap P^1 = \emptyset$ and $R^1 \cup P^1$ is a linear forest.

We would like to find a similar perfect matching R^2 of Q^2, that would join the perfect matching $R = R^1 \cup R^2$ of Q_d. However, we forbid some edges to be contained in R^2 which will preserve that $P \cup R$ is acyclic. The forbidden set of edges is

$$S = \left\{ xy \in E\bigl(K^2\bigr) \;\middle|\; \begin{array}{l} \exists x', y' \in V(Q^1) \text{ such that } xx', yy' \in P \text{ and} \\ \text{there exists a path from } x' \text{ to } y' \text{ of } P^1 \cup R^1 \end{array} \right\}.$$

Every vertex v of the graph $(V(K^1), P^1 \cup R^1)$ has degree one if and only if v is not covered by P^1. If there exists a path from x' to y' of $P^1 \cup R^1$ and $xx', yy' \in P$ and $xy \in E(K^2)$, then x' and y' are not covered by P^1 and x' and y' are vertices of both ends of a path of $P^1 \cup R^1$. Thus, the set of edges S is a matching of K^2. Moreover, the set of edges $P^2 \cup S$ is a matching of K^2 which is not perfect because S covers (not necessary all) vertices covered by P but not by P^2 and u_2 is not covered by P. Hence, there must exist a perfect matching R^2 of Q^2 by the induction such that $R^2 \cap (P^2 \cup S) = \emptyset$ and $R^2 \cup P^2 \cup S$ is a linear forest.

We show that the perfect matching $R = R^1 \cup R^2$ of Q_d satisfies the requirements of the lemma. For sake of contradiction, suppose that C is a cycle of $R \cup P$. Notice that C cannot belong to K^1 or to K^2. So C has edges in both K^1 and K^2. Now, we can shorten every path $xx' \cdots y'y$, such that $x, y \in V(Q^2)$, $x', y' \in V(Q^1)$, $xx', yy' \in P$ and $x' \cdots y'$ is a path of $P^1 \cup R^1$, by the edge $xy \in S$. Hence, we obtain a cycle of $R^2 \cup P^2 \cup S$, which is a contradiction. Thus, $P \cup R$ is a forest. Since every vertex in the graph $P \cup R$ has degree one or two, it is a linear forest. □

Riste Škrekovski [6] asked whether the following stronger form of Kreweras' conjecture could be true:

Does every (not necessarily perfect) matching of Q_d, $d \geqslant 2$, extends to a Hamilton cycle of Q_d?

The statement can be shown to be true for $d = 2, 3, 4$. However, our approach does not seem to lead to proving this stronger statement.

Acknowledgments

I am grateful to Riste Škrekovski for introducing Kreweras' conjecture to me during my visit to University of Ljubljana under the bilateral project BI-CZ/06-07-015 and MSMT-07-0405 between Slovenia and Czech Republic. I am also grateful to Martin Tancer for fruitful discussions.

References

[1] G. Kreweras, Matchings and Hamiltonian cycles on hypercubes, Bull. Inst. Combin. Appl. 16 (1996) 87–91.
[2] C. Savage, A survey of combinatorial Gray codes, SIAM Rev. 39 (1997) 605–629.
[3] L. Gros, Théorie du Baguenodier, Aimé Vingtrinier, Lyon, 1872.
[4] T. Dvořák, Hamiltonian cycles with prescribed edges in hypercubes, SIAM J. Discrete Math. 19 (2005) 135–144.
[5] D. Dimitrov, R. Škrekovski, T. Dvořák, P. Gregor, Gray Codes Faulting Matchings, manuscript.
[6] R. Škrekovski, private communication.